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A study of quasi-exactly solvable models within the quantum Hamilton–Jacobi formalism

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Abstract

A few quasi-exactly solvable models are studied within the quantum Hamilton–Jacobi formalism. By assuming a simple singularity structure of the quantum momentum function, we show that the exact quantization condition leads to the condition for quasi-exact solvability.

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1. Introduction

This paper reports an investigation of quasi-exactly solvable (QES) models within the quantum Hamilton–Jacobi (QHJ) formalism. One of the earliest investigations on QES models was done by Singh *et al* [1]. The QES models have been studied extensively, and for a general review and references we refer the reader to the book by Ushveridze [2]. In the recent years, a lot of interesting work has been done on QES potential models [3–13]. A complete list of canonical forms for the QES models has been obtained in [3, 4] and QES periodic potentials have been studied in [7–9]. The QES models have a characteristic property that, when the potential parameters satisfy a specified condition, analytic expressions for a few energy levels and their corresponding eigenfunctions can be obtained exactly. This condition between the parameters of the potential will be referred to as the quasi-exactly solvability condition.

A well-known example of a QES model is the sextic oscillator in one dimension, corresponding to the potential $V(x) = \alpha x^2 + \beta x^4 + \gamma x^6$. The condition of quasi-exactly solvability is found to be $\frac{1}{\sqrt{\gamma}}\left[\frac{\beta^2}{4\gamma} - \alpha\right] = 3 + 2n$, where n is a non-negative integer which is related to the number of levels for which exact energy eigenfunctions and eigenvalues can be computed. A number of other QES models have been constructed and studied within the algebraic and group theoretical approach. Interesting connections have been established between the sextic oscillator and second-order linear differential equations within a new approach to second-order linear differential equations [13].

In this paper we study the sextic oscillator and a few other QES models in one dimension. In each of these cases the condition of quasi-exactly solvability is derived within the QHJ approach. The next section contains an overview of the QHJ formalism and how it is used in the present paper to study the QES models. In the following four sections we investigate the sextic oscillator, the sextic oscillator with a centrifugal barrier, a hyperbolic potential and a circular potential. The last section contains our conclusions about exact solvability and quasi-exactly solvability.

2. Quantum Hamilton–Jacobi formalism

The QHJ formalism was initiated by Leacock and Padgett [14, 15] and was successfully applied to several exactly solvable models (ESM) in one dimension in quantum mechanics by Bhalla *et al* [16–18]. Our discussions in this paper will be limited to the QES problems in one dimension only. The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi + V(x)\psi = E\psi. \tag{1}$$

On substituting $\psi = \exp(\mathrm{i}S/\hbar)$ in (1) we obtain the following equation for S:

$$\left(\frac{\mathrm{d}S}{\mathrm{d}x}\right)^2 - \mathrm{i}\hbar\left(\frac{\mathrm{d}^2S}{\mathrm{d}x^2}\right) = 2m[E - V(x)]. \tag{2}$$

We define $p(x) = \frac{\mathrm{d}S}{\mathrm{d}x}$ which satisfies the Riccati equation

$$p^2(x) - \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}p(x) = 2m[E - V(x)]. \tag{3}$$

In the limit $\hbar \to 0$, (2) becomes the classical Hamilton–Jacobi equation and $p(x) \to p_{\mathrm{cl}}$ the classical momentum, which is

$$p_{\mathrm{cl}} = \sqrt{2m[E - V(x)]}. \tag{4}$$

Therefore, (2) and (3) will be referred to as QHJ equations, $p(x)$ will be called the quantum momentum function (QMF) and S will be the quantum action. In terms of the eigenfunction ψ of the energy, the QMF is given by

$$p(x) = -\mathrm{i}\hbar\frac{1}{\psi}\frac{\mathrm{d}\psi}{\mathrm{d}x}. \tag{5}$$

An important step in the QHJ formalism is to regard x as a complex variable and extend the definition of $p(x)$ to the complex plane. From (5) it is obvious that the zeros of the wavefunction correspond to the poles of the QMF. It is known that the bound state solutions of (1) corresponding to the nth level have n real zeros, correspondingly the QMF has n poles on the real line. From (3) it can be seen that if x is a point at which $V(x)$ is analytic and p has a pole, the pole must be of first order and the residue at that pole will be $-\mathrm{i}\hbar$. Therefore, the integral of the QMF taken along the contour C, which encloses these poles, will have the value $n\hbar$. Thus, we get

$$\frac{1}{2\pi}\oint_C p(x)\,\mathrm{d}x = n\hbar \tag{6}$$

and this is an exact quantization condition in one dimension [14, 15]. This exact quantization condition (6) has been used to obtain bound state energy levels without solving for the eigenfunctions for several ESM [16–18]. For this purpose one needs to know the location of singularities and their corresponding residues of the QMF in the complex plane. The QMF has two kinds of singularities, fixed and moving singularities. The fixed singularities correspond to the singularities of the potential and will be present in every solution of the Riccati equation, their location being independent of the initial conditions. The position of the moving singularities depends on the initial conditions. It is known that for the solutions of a Riccati equation only poles can appear as moving singularities. Therefore, if the potential is meromorphic, the solutions will also be meromorphic. Coming back to the QMF corresponding to the solutions of the Schrödinger equation, in addition to the poles, corresponding to the n real zeros of the nth excited state, in general there may be other moving poles. Knowledge about these poles is needed to apply the QHJ method. The residue at any fixed pole can be computed from (3). This, being a quadratic equation, will lead to two solutions and, therefore, a boundary condition on the QMF is needed to pick up the physical solutions. Leacock and Padgett proposed that one should make use of the condition

$$\lim_{\hbar\to 0} p(x) \to p_{\mathrm{cl}}. \tag{7}$$

From (4) it should be noted that the classical turning points become the branch points of p_{cl}. It has to be emphasized here that, when x is a complex variable, p_{cl} is a double-valued function. A full definition of p_{cl} in the complex plane as a function of a complex variable x requires us to select a particular branch of p_{cl} which is assumed to have a branch cut in the classical region and a positive value just below the branch cut. Locating the singular points of the QMF and imposing the boundary conditions discussed above are two crucial but usually difficult steps in arriving at the correct solutions. In the earlier studies of ESM, it was easy to guess the singularity structure of the QMF, and it was found that there were no moving poles away from the real axis. For the QES models studied in this paper, it is very difficult to find the location of the singularities of the QMF. In order to be able to make some progress, we make a simplifying assumption about the moving poles and the nature of the singularity at infinity. The details of moving and possible fixed poles will be given in each case separately as and when we discuss the model. We now state our main assumption, common to all models studied, *i.e., the point at infinity is an isolated point and that it is a pole of finite order and not an essential singularity*. Under these assumptions we show that imposing the exact quantization condition leads to the condition of QES in each case.

3. Sextic oscillator

The potential for the sextic oscillator is

$$V(x) = \alpha x^2 + \beta x^4 + \gamma x^6 \qquad \gamma > 0. \tag{8}$$

From now onwards we set $\hbar = 2m = 1$. For the nth exited state, QMF has n moving poles on the real axis and we assume that there are no other moving poles. In order to use the quantization condition (6), we need to evaluate the integral $J(E)$,

$$J(E) = \frac{1}{2\pi}\oint_C p(x)\,\mathrm{d}x \tag{9}$$

along the contour C. As a first step, it is convenient to deform the contour C to a large circular contour Γ, with the centre at the origin and large enough to enclose all the singularities of the QMF in the finite complex plane. This is possible because the point at infinity is assumed to be an isolated singularity and the value of the integral remains unchanged. Hence

$$J(E) = \frac{1}{2\pi}\oint_\Gamma p(x)\,\mathrm{d}x. \tag{10}$$

In order to evaluate the integral along the large circle Γ, we apply an inversion mapping $x \to y = \frac{1}{x}$. Under this mapping the circular contour Γ gets mapped to another circular contour γ in the y-plane. The only singular point inside this contour is the point $y = 0$, which is the image of the point at infinity in the x-plane. Thus, we obtain from (10)

$$J(E) = \frac{1}{2\pi} \oint_{\gamma} \tilde{p}(y) \frac{1}{y^2} \, \mathrm{d}y. \tag{11}$$

To evaluate the integral in (11) a Laurent expansion of $\tilde{p}(y)$ is made:

$$\tilde{p}(y) = \sum_{n=1}^{3} \frac{b_n}{y^n} + \sum_{n=1}^{\infty} a_n y^n. \tag{12}$$

Substituting this expansion in (11) and integrating term by term, we get

$$J(E) = \mathrm{i}a_1. \tag{13}$$

It only remains to compute the coefficient a_1 of the Laurent expansion given in (12). To do this we start from the QHJ equation

$$\tilde{p}^2(y) + \mathrm{i}y^2 \frac{\mathrm{d}}{\mathrm{d}y} \tilde{p}(y) = E - \frac{\alpha}{y^2} - \frac{\beta}{y^4} - \frac{\gamma}{y^6}. \tag{14}$$

Substituting the Laurent expansion and equating the coefficients of different powers of y in both sides of equation (14), we successively get the following equations:

$$b_3^2 = -\gamma \tag{15}$$

$$b_2 = 0 \tag{16}$$

$$2b_1 b_3 = -\beta \tag{17}$$

$$b_1^2 + b_3(2a_1 - 3\mathrm{i}) = -\alpha. \tag{18}$$

It is important to know that we would get two solutions for b_1 corresponding to the two solutions of $b_3 = \pm\mathrm{i}\sqrt{\gamma}$. This happens due to the fact that the QHJ is quadratic in the QMF. Thus one needs a boundary condition to pick the correct solution. We propose to use the square integrability of the wavefunction instead of the original boundary condition, explained in the introduction, which was proposed by Leacock and Padgett. This is because the original boundary condition is difficult to implement in the present case due to the presence of six branch points in the p_{cl}. In order to find the restrictions coming from the square integrability, we compute the wavefunction

$$\psi(x) = \exp\left(\int \mathrm{i}p(x) \, \mathrm{d}x\right) \tag{19}$$

for large x as follows. The most important term in the Laurent expansion (12) for small $y \approx 0$, corresponding to large x, is

$$\tilde{p}(y) \approx \frac{b_3}{y^3} \tag{20}$$

and the wavefunction for large x becomes

$$\psi(x) \approx \exp\left(\mathrm{i}\frac{b_3 x^4}{4}\right) \mathrm{d}x. \tag{21}$$

Out of the two solutions, $b_3 = \pm\mathrm{i}\sqrt{\gamma}$, ψ is square integrable only for $b_3 = \mathrm{i}\sqrt{\gamma}$. Using this value of b_3 and calculating $J(E)$ from (13), the quantization condition gives

$$\frac{1}{\sqrt{\gamma}}\left(\frac{\beta^2}{4\gamma} - \alpha\right) = 3 + 2n. \tag{22}$$

In order to compare the result in (22) with the known condition [2], we write

$$\gamma = a^2 \qquad \beta = 2ab. \tag{23}$$

Thus, we get $\alpha = b^2 - a(3 + 2n)$ which agrees with the known result.

4. Sextic oscillator with the centrifugal barrier

This kind of model was first studied by Houtot [19]. The potential we discuss here is

$$V(x) = 4\left(S - \frac{1}{4}\right)\left(S - \frac{3}{4}\right)\frac{1}{x^2} + \left[b^2 - 4a\left(S + \frac{1}{2} + M\right)\right]x^2 + 2abx^4 + a^2x^6 \tag{24}$$

and the range of S is taken to be $4S > 3$. We note that the potential goes to ∞ as $x \to 0$ and $x \to \infty$. Hence, the classical turning points will be on the positive real axis. The physical motion in both classical and quantum situations will be confined to the positive real axis only. As has been discussed earlier, there will be n moving poles in the classical region which are enclosed by the contour C in the quantization condition stated in (6). When we extend the definition of x, to take all the complex values, we expect n additional moving poles on the negative real axis. These come from the symmetry of the potential under the transformation $x \to -x$. Just as in the quantization condition (6), we have

$$\frac{1}{2\pi}\oint_{C_1} p(x)\,\mathrm{d}x = n \tag{25}$$

where C_1 is the contour which encloses the n additional moving poles on the negative real axis. The QHJ equation for this potential is

$$\begin{aligned} p^2(x) - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}p(x) = E - 4\left(S - \frac{1}{4}\right)\left(S - \frac{3}{4}\right)\frac{1}{x^2} \\ - \left[b^2 - 4a\left(S + \frac{1}{2} + M\right)\right]x^2 - 2abx^4 - a^2x^6. \end{aligned} \tag{26}$$

We observe that in addition to the $2n$ moving poles, $x = 0$ is a fixed pole. In this case we assume that there are no other singularities in the finite complex plane. As in the case of the sextic oscillator, in order to evaluate $J(E)$ in (9), we deform the contour C to a large circular contour Γ, which will enclose the $2n$ moving poles and the fixed pole at $x = 0$. Therefore, we have

$$\oint_\Gamma p(x)\,\mathrm{d}x = \oint_{\gamma_0} p(x)\,\mathrm{d}x + \oint_C p(x)\,\mathrm{d}x + \oint_{C_1} p(x)\,\mathrm{d}x \tag{27}$$

where γ_0 is the contour enclosing only the fixed pole at $x = 0$. Expanding $p(x)$ in the Laurent series, in powers of x

$$p(x) = \frac{b_1}{x} + \sum_{n=0}^{\infty} a_n x^n \tag{28}$$

we get

$$\frac{1}{2\pi}\oint_{\gamma_0} p(x)\,\mathrm{d}x = \mathrm{i}b_1. \tag{29}$$

We fix b_1 by substituting the Laurent series in (26). Thus, we obtain

$$b_1 = \frac{\mathrm{i}}{2}[4S - 3] \qquad b_1 = -\frac{\mathrm{i}}{2}[4S - 1]. \tag{30}$$

The choice for the value of b_1 consistent with square integrability in the specified range is

$$b_1 = -\frac{\mathrm{i}}{2}[4S - 1]. \tag{31}$$

Thus, (27) becomes

$$\frac{1}{2\pi}\oint_\Gamma p(x)\,\mathrm{d}x = \mathrm{i}b_1 + 2n. \tag{32}$$

The integral along Γ is computed by changing the variable from $x \to y = \frac{1}{x}$ and proceeding in the same way as was done for the sextic oscillator. From (31) we then obtain

$$M = n \tag{33}$$

which is the quasi-exact solvability condition for the sextic oscillator with the centrifugal barrier.

5. Circular potential

The potential for this model is

$$V(x) = \frac{A}{\sin^2 x} + \frac{B}{\cos^2 x} - C\sin^2 x + D\sin^4 x \tag{34}$$

where

$$A = 4\left(S_1 - \tfrac{1}{4}\right)\left(S_1 - \tfrac{3}{4}\right) \qquad B = 4\left(S_2 - \tfrac{1}{4}\right)\left(S_2 - \tfrac{3}{4}\right) \tag{35}$$

$$C = q_1^2 + 4q_1(S_1 + S_2 + M) \qquad D = q_1^2 \tag{36}$$

and the ranges of S_1 and S_2 are $2S_1 > 1$ and $2S_2 > 1$, respectively. The QHJ equation is

$$p^2(x) - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}p(x) = E - \frac{A}{\sin^2 x} - \frac{B}{\cos^2 x} + C\sin^2 x - D\sin^4 x \tag{37}$$

and the quantization condition is

$$\frac{1}{2\pi}\oint_C p(x)\,\mathrm{d}x = n. \tag{38}$$

Defining $\sin^2 x = t$, the QHJ equation becomes

$$\tilde{p}^2(t) - 2\mathrm{i}\sqrt{t(1-t)}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{p}(t) = E - \frac{A}{t} - \frac{B}{1-t} + Ct - Dt^2 \tag{39}$$

where $\tilde{p}(t) \equiv p(x)$. Defining q by $\tilde{p} = \sqrt{t(1-t)}q$ we obtain the QHJ equation in t variable as follows:

$$q^2 - 2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}q - \frac{\mathrm{i}(1-2t)q}{t(1-t)} = \frac{E}{t(1-t)} - \frac{A}{t^2(1-t)} - \frac{B}{t(1-t)^2} + \frac{C}{1-t} - \frac{Dt}{1-t}. \tag{40}$$

The quantization condition (38) becomes

$$\frac{1}{2\pi}\oint_C \frac{q(t)}{2}\,\mathrm{d}t = n. \tag{41}$$

We observe that QMF has fixed poles at $t = 1, 0$ and proceeding in a way similar to section 4, we obtain the quasi-exact solvability condition as

$$M = n. \tag{42}$$

6. Hyperbolic potential

The QHJ equation for this potential is

$$p^2(x) - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}p(x) = E + \frac{A}{\cosh^2 x} - \frac{B}{\sinh^2 x} + C\cosh^2 x - D\cosh^4 x \tag{43}$$

where

$$A = 4\left(S_1 - \tfrac{1}{4}\right)\left(S_1 - \tfrac{3}{4}\right) \qquad B = 4\left(S_2 - \tfrac{1}{4}\right)\left(S_2 - \tfrac{3}{4}\right) \tag{44}$$

$$C = q_1^2 + 4q_1(S_1 + S_2 + M) \qquad D = q_1^2 \tag{45}$$

and the ranges of S_1 and S_2 are $2S_1 > 1$ and $2S_2 > 1$, respectively. A change of variable from $x \to t = \cosh x$ changes $p(x) \to \tilde{p}(t) = p(\cosh^{-1} x)$, then defining

$$\tilde{p}(t) = \sqrt{t^2 - 1}q(t). \tag{46}$$

The QHJ equation becomes

$$q^2 - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}q - \frac{\mathrm{i}tq}{t^2 - 1} = \frac{E}{t^2 - 1} + \frac{A}{t^2(t^2 - 1)} - \frac{B}{(t^2 - 1)^2} + \frac{Ct^2}{t^2 - 1} - \frac{q_1^2 t^4}{t^2 - 1}. \tag{47}$$

The quantization condition is

$$\frac{1}{2\pi}\oint q(t)\,\mathrm{d}t = n. \tag{48}$$

We expect $2n$ moving poles on the entire real line as in section 4, and fixed poles at $t = 0, \pm 1$. These poles together with a pole of finite order at infinity lead to the condition

$$M = n \tag{49}$$

which agrees with the known condition of QES of this model.

7. Conclusions

In the limit $\hbar \to 0$ the QMF of the sextic oscillator goes to p_{cl} which has six branch points. Therefore, in general we expect a complicated singularity structure for the sextic and also for other models studied in this paper. In the analysis presented in the previous sections it was assumed that there are no moving poles off the real axis. A closer look at the derivation of the condition of quasi-exact solvability shows that the assumption, 'no moving poles of the QMF off the real axis' can be replaced by a weaker assumption, 'the QMF has a finite number of moving poles in the complex plane', without altering any of the results. In fact, it can be seen from explicit solutions in [2, 4] that the algebraic eigenfunctions do have complex zeros. For all the models that have been studied here, the algebraic part of the spectrum and eigenfunctions are well known and in all cases the QMF has a pole at infinity. Thus, we conclude that, for the class of potentials studied here, QES models are the only models for which the QMF has a pole at infinity and a finite number of moving poles in the complex plane. For each model the known algebraic eigenfunctions correspond to the QMF having singularity structure as postulated. For completeness it must be mentioned that the assumption of a finite number of poles is not independent of the assumption that the point at infinity is a pole. For a large class of potentials which are analytic everywhere, except for isolated singularities, the moving singularities of solutions of the QHJ can only be poles. An infinite number of such poles will, therefore, have an accumulation point at infinity and making it $z = \infty$, a non-isolated essential singular point.

The parameter n that appears in the exact quantization condition is related to the number of moving poles in the QMF and has different roles to play for the ESM and QES models. In the ESM each value of n corresponds to an energy level and an eigenfunction with n real zeros. In the case of the QES models, n appears as a parameter in the expression for the potential and picks out a particular QES model within a family of potentials; varying n gives rise to a different potential within the family. Recalling that n also determines the number of moving poles of the QMF, it appears reasonable to expect that all the algebraic eigenfunctions for given QES models (fixed n) will have the same number of complex zeros determined by n. An explicit check reveals that this expectation is true for the sextic oscillator. A preliminary study reveals that the poles of the QMF for QES periodic potentials have a richer structure. A detailed study of the location of the poles of the QMF in different QES models will be reported elsewhere.

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